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Amendments of a Stochastic Restricted Principal Components Regression Estimator in the Linear Model

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Abstract: Principal component Analysis (PCA) is one of the popular methods used to solve the multicollinearity problem. Researchers in 2014 proposed an estimator to solve this problem in the linear model when there were stochastic linear restrictions on the regression coefficients. This estimator was called the stochastic restricted principal components (SRPC) regression estimator. The estimator was constructed by combining the ordinary mixed estimator (OME) and the principal components regression (PCR) estimator. It ignores the number of components (orthogonal matrix T_r) that the researchers choose to solve the multicollinearity problem in the data matrix (X). This paper proposed four different methods (Lagrange function, the same technique, the constrained principal component model, and substitute in model) to modify the (SRPC) estimator to be used in case of multicollinearity. Finally, a numerical example, an application, and simulation study have been introduced to illustrate the performance of the proposed estimator.

Keywords: Constrained Principal Components Analysis, General Linear Model, Principal Component Analysis, Simulation and Application, Stochastic Restricted Principal Components

1. Introduction

According to the Gauss Markov theorem, the linear regression model (LM) take the form:

$$Y_{n.1} = X_{n.p}\beta_{p.1} + \epsilon_{n.1}[1] \tag{1}$$

where Y is an $n \times 1$ vector of responses, X is an $n \times p$ observed matrix of the variables, assumed to have full rank, i.e., rank $(X) = p$, β is a $p \times 1$ vector of unknown parameters, and ϵ is an $n \times 1$ vector of error terms assumed to be multivariate normally distributed with mean 0 and variance covariance $\sigma^2 I_p$. It is known that the ordinary least squares (OLS) estimator of β is:

$$\hat{\beta}_{OLS} = (X^/X)^{-1}X^/Y \tag{2}$$

$\hat{\beta}_{OLS}$ is normally distributed $\mathcal{N}(\beta, \sigma^2(X^/X)^{-1})$. The standard regression model assumes that the column vectors in X are linearly independent. The restricted model for $\hat{\beta}_{OLS}$ can be written as $k = R\beta$ where R is an q x p matrix ($q \le p$), and k is q x 1 vector of restrictions. The restricted estimator $\hat{\beta}_{OLS}^{C}$ using Lagrange function was derived as follows:

$$L = (Y - X\beta)^/(Y - X\beta) + \lambda(k - R\beta_{OLS})$$

$$\frac{\partial}{\partial\beta}L = -2X^/Y + 2(X^/X)\,\beta_{OLS}^{C} + R^/\lambda = 0$$

$$\frac{\partial}{\partial\lambda}L = k - R\beta_{OLS}^{C} = 0$$

$$\lambda = -2(R(X^/X)^{-1}R^/)^{-1}(k - R\hat{\beta}_{OLS})$$

$$\hat{\beta}_{OLS}^{C} = \hat{\beta}_{OLS} + (X^/X)^{-1}R^/(R(X^/X)^{-1}R^/)^{-1}(k - R\hat{\beta}_{OLS}) \tag{3}$$

Researchers in 1961 used the next method to get the Ordinary Mixed Estimator (OME) for the least squares method, where they combined between the LM and the restricted model as follows [2]:

$$\binom{Y}{k} = \binom{X}{R}\beta + \binom{\epsilon}{\epsilon^*} \tag{4}$$

where: $\mathbb{E}\left\{\binom{\epsilon}{\epsilon^*}(\epsilon^/ \quad \epsilon^{*/})\right\} = \sigma^2\begin{pmatrix} I & 0 \\ 0 & V\end{pmatrix}$, i.e. $\sigma^2 V$ is the variance of the error term that was found in the restricted model ($var\ (\epsilon^*)$), where V assumed to be known and positive definite (pd) matrix. The (OME) for the least square is given by equations (5) and (6) which were equivalent as follows:

$$\hat{\beta}_{OME} = (X^{/}X + R^{/}V^{-1}R)^{-1}(X^{/}Y + R^{/}V^{-1}k) \quad (5)$$

$$= \hat{\beta}_{OLS} + (X^{/}X)^{-1}R^{/}(V + R(X^{/}X)^{-1}R^{/})^{-1}(k - R\hat{\beta}_{OLS}) \quad (6)$$

The expectation of the OME was $\mathbb{E}(\hat{\beta}_{OME}) = \beta$, the variance was given by $var(\hat{\beta}_{OME}) = \sigma^2(X^{/}X + R^{/}V^{-1}R)^{-1}$ [2], and the matrix mean square error $\text{MMSE}(\hat{\beta}_{OME}) = \sigma^2(X^{/}X + R^{/}V^{-1}R)^{-1}$. From (3) and (6), the equations indicate that $\hat{\beta}_{OME} = \hat{\beta}^{C}_{OLS}$ adding V to the term $R(X^{/}X)^{-1}R^{/}$

Section two presented another view of the SRPC, while section three introduced four different methods for computing the SRPC estimator. Finally, the last section introduced a numerical example to show the difference between the old method that introduced by previous papers [3], and the new method that was proposed in this paper.

2. Another View of the $\hat{\boldsymbol{\beta}}_{SRPC}$ Estimator

As indicated in 2014, the SRPC calculated the OME for principal component [3]. Unlike the estimator introduced by researchers in 1961 that calculated the OME for the least squares method [2]. They used (7) and (8) to derive their estimator:

$$k = RT_rT_r^{/}\beta + \epsilon^* \quad (7)$$

$$\binom{Y}{k} = \binom{X}{R}T_rT_r^{/}\beta + \binom{\epsilon}{\epsilon^*} \quad (8)$$

where $T_r = (t_1, t_2, \ldots, t_r)$ represents the remaining columns of the orthogonal matrix $T = (t_1, t_2, \ldots, t_p)$ after having deleted the last $p - r$ columns, where $0 \le r \le p$.

In a study [3] researchers assumed that all changes led to β in case of using principal component analysis. They used the equation: $\hat{\beta}_{PC} = T_rT_r^{/}\beta$ in their analysis, the summarizing of their study was as follows:

$$\hat{\beta}_{SRPC} = (X^{/}X + R^{/}V^{-1}R)^{-1}(T_rT_r^{/}X^{/}Y + R^{/}V^{-1}k) \quad (9)$$

$$\hat{\beta}_{SRPC} = \hat{\beta}_{PC} + (X^{/}X)^{-1}R^{/}(V + R(X^{/}X)^{-1}R^{/})^{-1}(k - R\hat{\beta}_{PC}) \quad (10)$$

where the expectation for their estimator was: $\mathbb{E}(\hat{\beta}_{OME}) = \beta + ((X^{/}X) + R^{/}V^{-1}R)^{-1}(T_rT_r^{/} - I)X^{/}X\beta$, while the variance was: $var(\hat{\beta}_{OME}) = \sigma^2(X^{/}X + R^{/}V^{-1}R)^{-1}(T_rT_r^{/}X^{/}XT_rT_r^{/} + R^{/}V^{-1}R)(X^{/}X + R^{/}V^{-1}R)^{-1}$, and the matrix mean square error take the form:

$$\text{MMSE}(\hat{\beta}_{OME}) = \sigma^2(X^{/}X + R^{/}V^{-1}R)^{-1}(T_rT_r^{/}X^{/}XT_rT_r^{/} + R^{/}V^{-1}R)(X^{/}X + R^{/}V^{-1}R)^{-1} + \left\{\left[((X^{/}X) + R^{/}V^{-1}R)^{-1}(T_rT_r^{/} - I)X^{/}X\beta\right]\left[((X^{/}X) + R^{/}V^{-1}R)^{-1}(T_rT_r^{/} - I)X^{/}X\beta\right]^{/}\right\}$$

In case of using the model shown in (4) in the principal component model, a new estimator is found. This new estimator is different from the estimator shown in (10). Researchers depended on the equations: $Y = X\ \beta + \epsilon$ and $k = R\beta + \epsilon^*$ to calculate expectation and variance for their estimator [3], and they ignored the principal component model and its assumptions $Y_{n.1} = X_{n.r}T_{r.P}\ T^{/}_{P.r}\ \beta_{r.1} + \epsilon_{n.1}$, $k_{q.1} = R_{q.r}T_{r.P}\ T^{/}_{P.r}\ \beta_{r.1} + \epsilon^*{}_{q.1}$. If they had taken these assumptions in to consideration, the expectation and the variance would change. The expectation would become: $\mathbb{E}(\hat{\beta}_{SRPC}) = \beta_{pc}$ i.e their estimator would be unbiased for the principal component parameter β_{pc}. The researchers should use the estimator of principal component when they want to calculate the OME for principal component (SRPC) and don't use OLS in this case [4]. This paper tries to solve these problems by introducing the same estimator using the principal component model, where the next sections indicate four different methods of the proposed estimator.

3. The Proposed Estimator

The author don't agree with researchers in 2014 at equations (9) and (10), where they used the principal component method to solve the problem of Multicollinearity. In case of highly correlated between variables (Multicollinearity) the matrix $(X^{/}X)^{-1}$ doesn't exist, so the researchers cannot estimate $\hat{\beta}_{SRPC}$]5 [A new estimator has been proposed using the following equations by four different methods.

$$\binom{Y}{k} = \binom{XT_r}{RT_r}T_r^{/}\beta + \binom{\epsilon}{\epsilon^*} \quad (11)$$

where: $\mathbb{E}(\epsilon\epsilon^{*/}) = 0$, and:

$$\mathbb{E}\left\{\binom{\epsilon}{\epsilon^*}(\epsilon^{/} \quad \epsilon^{*/})\right\} = \sigma^2\begin{pmatrix} I & 0 \\ 0 & V \end{pmatrix} \quad (12)$$

Theorem: $\hat{\beta}_{MSRPC} = \hat{\beta}_{PC} + T_r\Lambda_r^{-1}T_r{}^{/}R^{/}(RT_r\Lambda_r^{-1}T_r{}^{/}R^{/} + V)^{-1}(k - R\hat{\beta}_{PC})$, this theorem can be proved by four methods: 1. Lagrange function, 2. the same technique which used by researchers in 1961 to get the OME estimator [2], 3. the constrained principal component model, and 4. substitute in model (4) using principal component assumptions.

3.1. The First Method

According to previous studies, in case of multicollinearity problem, the researchers used another forms to estimate the parameters like principal component regression PCR, where this problem occurs when the predictors included in the linear model are highly correlated with each other. In this case the matrix $X^{/}X$ tends to be singular hence, identifying the least squares estimators will face numerical problems [6].

The researchers used the orthogonal matrix T in the GLM to get the PCR estimator for β as follows:

$$Y_{n.1} = X_{n.p} T_{P.P}\, T_{P.P}^{/}\, \beta_{p.1} + \epsilon_{n.1} \quad (13)$$

A spectral decomposition of the matrix $X^{/}X$ was given by:

$$X^{/}X = \left(T_r, T_{p-r}\right)\begin{pmatrix}\Lambda_r & 0\\ 0 & \Lambda_{p-r}\end{pmatrix}\begin{pmatrix}T_r\\ T_{p-r}\end{pmatrix} \quad (14)$$

where $\Lambda_r = T_r^{/}X^{/}XT_r$ is diagonal matrix such that the main diagonal elements are the r largest eigenvalues of $X^{/}X$, while the main diagonal elements of the Λ_{p-r} matrix are the remaining $p - r$ eigenvalues.

The PCR estimator for β can be written as:

$$\hat{\beta}_{PC} = T_r\,(T_r^{/}X^{/}XT_r)^{-1}\,T_r^{/}X^{/} \quad (15)$$

The expectation of $\hat{\beta}_{PC}$ is $(\hat{\beta}_{PC}) = T_rT_r{}^{/}\beta = \beta_{PC}$, $var(\hat{\beta}_{PC}) = \sigma^2 T_r(\Lambda_r)^{-1}T_r{}^{/}$, and the matrix mean square error is $\mathrm{MMSE}(\hat{\beta}_{PC}) = \sigma^2 T_r(\Lambda_r)^{-1}T_r{}^{/}$.

The restricted estimator $\hat{\beta}_{PC}^{C}$ using Lagrange function is given by:

$$L = (Y - XT_rT_r{}^{/}\beta)^{/}(Y - XT_rT_r{}^{/}\beta) + \lambda^{/}(k - RT_rT_r{}^{/}\beta)$$

$$= (Y^{/} - \beta^{/}T_rT_r{}^{/}X^{/})\,(Y - XT_rT_r{}^{/}\beta) + \lambda^{/}(k - RT_rT_r{}^{/}\beta)$$

$$= Y^{/}Y - Y^{/}XT_rT_r{}^{/}\beta \; - \beta^{/}T_rT_r{}^{/}X^{/}Y + \beta^{/}T_rT_r{}^{/}X^{/}XT_rT_r{}^{/}\beta \; + \lambda^{/}(k - RT_rT_r{}^{/}\beta)$$

$$\frac{\partial}{\partial\beta}L = -Y^{/}XT_rT_r{}^{/} - T_rT_r{}^{/}X^{/}Y + 2T_rT_r{}^{/}X^{/}XT_rT_r{}^{/}\,\beta_{OLS}^{C} - T_rT_r{}^{/}R^{/}\lambda = 0$$

$$= -2T_rT_r{}^{/}X^{/}Y + 2T_rT_r{}^{/}(X^{/}X)\,\beta_{OLS}^{C} - T_rT_r{}^{/}R^{/}\lambda = 0 \quad (16)$$

$$\frac{\partial}{\partial\lambda}L = k - RT_rT_r{}^{/}\beta_{OLS}^{C} = k - R\beta_{PC}^{C} = 0$$

$$\therefore k = RT_rT_r{}^{/}\beta_{OLS}^{C}$$

From (16):

$$T_rT_r{}^{/}R^{/}\lambda = -2T_rT_r{}^{/}X^{/}Y + 2T_rT_r{}^{/}(X^{/}X)\,T_rT_r{}^{/}\beta_{OLS}^{C}$$

$$RT_rT_r{}^{/}(T_rT_r{}^{/}(X^{/}X)\,T_rT_r{}^{/}\,)^{-1}T_rT_r{}^{/}R^{/}\lambda = -2RT_rT_r{}^{/}(T_rT_r{}^{/}(X^{/}X)T_rT_r{}^{/}\,)^{-1}T_rT_r{}^{/}X^{/}Y + 2R\,T_rT_r{}^{/}\beta_{OLS}^{C}.$$

$$RT_r(T_r{}^{/}(X^{/}X)\,T_r\,)^{-1}T_r{}^{/}R^{/}\lambda = -2RT_r(T_r{}^{/}\,(X^{/}X)\,T_r\,)^{-1}T_r{}^{/}X^{/}Y + 2k$$

$$RT_r(T_r{}^{/}(X^{/}X)\,T_r\,)^{-1}T_r{}^{/}R^{/}\lambda = 2(k - R\hat{\beta}_{PC})$$

$$\therefore \lambda = 2(RT_r(T_r{}^{/}X^{/}XT_r)^{-1}T_r{}^{/}R^{/})^{-1}(k - R\hat{\beta}_{PC})$$

Substitute in (16):

$$T_rT_r{}^{/}R^{/}\lambda = -2T_rT_r{}^{/}X^{/}Y + 2T_rT_r{}^{/}(X^{/}X)\,T_rT_r{}^{/}\beta_{OLS}^{C}$$

$$2T_rT_r{}^{/}R^{/}(RT_r(T_r{}^{/}X^{/}XT_r)^{-1}T_r{}^{/}R^{/})^{-1}(k - R\hat{\beta}_{PC}) = -2T_rT_r{}^{/}X^{/}Y + 2T_rT_r{}^{/}(X^{/}X)\,T_rT_r{}^{/}\beta_{OLS}^{C}$$

$$T_rT_r{}^{/}(X^{/}X)\,T_rT_r{}^{/}\beta_{OLS}^{C} = T_rT_r{}^{/}R^{/}(RT_r(\Lambda)^{-1}T_r{}^{/}R^{/})^{-1}(k - R\hat{\beta}_{PC}) + T_rT_r{}^{/}X^{/}Y$$

$$\hat{\beta}_{PC}^{C} \;=\; (T_rT_r{}^{/}X^{/}XT_rT_r{}^{/})^{-1}\; T_rT_r{}^{/}R^{/}(RT_r(\Lambda)^{-1}T_r{}^{/}R^{/})^{-1}(k - R\hat{\beta}_{PC}) + \;(T_rT_r{}^{/}X^{/}XT_rT_r{}^{/})^{-1}\,T_rT_r{}^{/}X^{/}Y$$

$$=\; T_r(T_r{}^{/}X^{/}XT_r)^{-1}\; T_r{}^{/}T_rT_r{}^{/}R^{/}(RT_r(\Lambda)^{-1}T_r{}^{/}R^{/})^{-1}(k - R\hat{\beta}_{PC}) \; + T_r(T_r{}^{/}X^{/}XT_r)^{-1}\; T_r{}^{/}T_rT_r{}^{/}X^{/}Y$$

$$= T_r(\Lambda_r\,)^{-1}\; T_r{}^{/}R^{/}(RT_r(\Lambda_r)^{-1}T_r{}^{/}R^{/})^{-1}(k - R\hat{\beta}_{PC}) + T_r(\Lambda_r\,)^{-1}\; T_r{}^{/}X^{/}$$

$$\hat{\beta}_{PC}^{C} = \hat{\beta}_{PC} + T_r\Lambda_r^{-1}T_r{}^{/}R^{/}(RT_r\Lambda_r^{-1}T_r{}^{/}R^{/})^{-1}(k - R\hat{\beta}_{PC}) \quad (17)$$

By adding the term V to $(RT_r\Lambda_r^{-1}T_r{}^{/}R^{/})$:

$$\hat{\beta}_{MSRPC} = \hat{\beta}_{PC} + T_r\Lambda_r^{-1}T_r{}^{/}R^{/}(RT_r\Lambda_r^{-1}T_r{}^{/}R^{/} + V)^{-1}(k - R\hat{\beta}_{PC}) \quad (18)$$

where $(MSRPC)$ refers to the modified stochastic restricted principal components.

3.2. The Second Method

Researchers in 2009 said that $\hat{\beta}_{PC} = T_r\,(\Lambda_r)^{-1}\,T_r^{/}X^{/}Y,$ where $\Lambda_r = T_r^{/}X^{/}XT_r$ [6]. Following previous studies [2], if $\hat{\beta}_{OLS}$ changed to $\hat{\beta}_{PC}$, $\hat{\beta}_{MSRPC}$ become as follows:

$$\hat{\beta}_{MSRPC} = T_r(\Lambda_r + T_r^{/}R^{/}V^{-1}RT_r)^{-1}(T_r^{/}X^{/}Y + T_r^{/}R^{/}V^{-1}k) \quad (19)$$

$$= (\Lambda_r T_r^{/} + T_r^{/}R^{/}V^{-1}RT_rT_r^{/})^{-1}(T_r^{/}X^{/}XT_rT_r^{/}\beta + T_r^{/}X^{/}\epsilon + T_r^{/}R^{/}V^{-1}RT_rT_r^{/}\beta + T_r^{/}R^{/}V^{-1}\epsilon^{*})$$

$$= (\Lambda_r T_r^{/} + T_r^{/}R^{/}V^{-1}RT_rT_r^{/})^{-1}(\Lambda_r T_r^{/}\beta + T_r^{/}R^{/}V^{-1}RT_rT_r^{/}\beta) + (\Lambda_r T_r^{/} + T_r^{/}R^{/}V^{-1}RT_rT_r^{/})^{-1}(T_r^{/}X^{/}\epsilon + T_r^{/}R^{/}V^{-1}\epsilon^{*})$$

$$= \hat{\beta}_{PC} + (\Lambda_r T_r^{/} + T_r^{/}R^{/}V^{-1}RT_rT_r^{/})^{-1}(T_r^{/}X^{/}\epsilon + T_r^{/}R^{/}V^{-1}\epsilon^{*})$$

$$\mathbb{E}(\hat{\beta}_{MSRPC}) = \beta_{P.C} \quad (20)$$

$$\mathbb{E}\left\{(\hat{\beta}_{MSRPC} - \beta_{PC})\,(\hat{\beta}_{MSRPC} - \beta_{PC})^{/}\right\}$$

$$= T_r(\Lambda_r + T_r^{/}R^{/}V^{-1}RT_r)^{-1}(T_r^{/}X^{/}\mathbb{E}(\epsilon\epsilon^{/})XT_r + T_r^{/}R^{/}V^{-1}\mathbb{E}(\epsilon^{*}\epsilon^{*/})V^{-1}RT_r)(\Lambda_r + T_r^{/}R^{/}V^{-1}RT_r)^{-1}T_r^{/}$$

$$= \sigma^2 T_r(\Lambda_r + T_r^{/}R^{/}V^{-1}RT_r)^{-1}(T_r^{/}X^{/}XT_r + T_r^{/}R^{/}V^{-1}RT_r)(\Lambda_r + T_r^{/}R^{/}V^{-1}RT_r)^{-1}T_r^{/}$$

$$var(\hat{\beta}_{MSRPC}) = \sigma^2 T(\Lambda_r + T_r^{/}R^{/}V^{-1}RT_r)^{-1}T_r^{/} \quad (21)$$

Using equation (19), equation (22) can be proved knowing that:
$(A + BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + CA^{-1}B)^{-1}DA^{-1}$, where $A, B, C,$ and D are pd matrices [7].

$$\hat{\beta}_{MSRPC} = T_r(\Lambda_r + T_r^{/}R^{/}V^{-1}RT_r)^{-1}(T_r^{/}X^{/}Y + T_r^{/}R^{/}V^{-1}k)$$

$$= T_r({\Lambda_r}^{-1} - {\Lambda_r}^{-1}T_r^{/}R^{/}(V + RT_r{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1}RT_r{\Lambda_r}^{-1})\,(T_r^{/}X^{/}Y + T_r^{/}R^{/}V^{-1}k)$$

$$= T_r({\Lambda_r}^{-1}T_r^{/}X^{/}Y + {\Lambda_r}^{-1}T_r^{/}R^{/}V^{-1}k - {\Lambda_r}^{-1}T_r^{/}R^{/}(V + RT_r{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1}\,RT_r{\Lambda_r}^{-1}\,T_r^{/}X^{/}Y$$
$$- {\Lambda_r}^{-1}T_r^{/}R^{/}(V + RT_r{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1}\,RT_r{\Lambda_r}^{-1}\,T_r^{/}R^{/}V^{-1}k)$$

$$= \hat{\beta}_{PC} + T_r{\Lambda_r}^{-1}T_r^{/}R^{/}V^{-1}k - T_r{\Lambda_r}^{-1}T_r^{/}R^{/}(V + RT_r{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1}\,R\,\hat{\beta}_{PC}$$
$$- T_r{\Lambda_r}^{-1}T_r^{/}R^{/}(V + RT_r{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1}\,RT_r{\Lambda_r}^{-1}\,T_r^{/}R^{/}V^{-1}k)$$

$$= \hat{\beta}_{PC} + T_r{\Lambda_r}^{-1}T_r^{/}R^{/}\left[\,V^{-1}k - (V + RT_r{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1}R\hat{\beta}_{PC} - (V + RT_r{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1}RT_r{\Lambda_r}^{-1}T_r^{/}R^{/}V^{-1}k\right]$$
$$= \hat{\beta}_{PC}$$
$$+ T_r{\Lambda_r}^{-1}T_r^{/}R^{/}\left[-(V + RT_r{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1}R\hat{\beta}_{PC} + (V^{-1} - (V + RT_r{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1}RT_r{\Lambda_r}^{-1}T_r^{/}R^{/}V^{-1})^{-1}k\right]$$
$$= \hat{\beta}_{PC}$$
$$+ T_r{\Lambda_r}^{-1}T_r^{/}R^{/}\left[-(V + RT_r{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1}R\hat{\beta}_{PC} + (V^{-1} - V^{-1}\,(I\right.$$
$$\left.+ RT_r{\Lambda_r}^{-1}T_r^{/}R^{/}V^{-1})^{-1}RT_r{\Lambda_r}^{-1}T_r^{/}R^{/}V^{-1})^{-1}k\right] + \left[V^{-1} - V^{-1}\,(RT_r{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1} + V^{-1}\right]k$$

Note that:

$$V^{-1} - V^{-1}\,(RT_r{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1} + V^{-1} = V^{-1} - V^{-1}\,RT_r{\Lambda_r}^{-1}T_r^{/}R^{/}\,(I + V^{-1}RT_r{\Lambda_r}^{-1}T_r^{/}R^{/}\,) = (V + RT{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1}$$

$$= \hat{\beta}_{PC} + T_r{\Lambda_r}^{-1}T_r^{/}R^{/}\left[-(V + RT{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1}R\hat{\beta}_{pc} + (V + RT{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1}k\right]$$

Then,

$$\hat{\beta}_{MSRPC} = \hat{\beta}_{PC} + T_r{\Lambda_r}^{-1}T_r^{/}R^{/}(V + RT_r{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1}(k - R\hat{\beta}_{PC}) \quad (22)$$

3.3. The Third Method

According to previous studies, Constrained Principal Component Analysis is a method for structural analysis of multivariate data that combine features of regression analysis with principal component analysis. In this method, the original data are first decomposed into several components according to external information. The components are then subjected to principal component analysis to explore structures within the components [8].

The constrained principal component model is:

$$Z_{N.n} = G_{N.p}M_{p.q}H_{q.n}^{/} + B_{N.q}H_{q.n}^{/} + G_{N.p}C_{p.n} + E_{N.n} \quad (23)$$

where Z is an $N \times n$ matrix of responses, G and H are observed matrices of the variables, assumed to have full rank,

$M, B,$ and C are matrices of unknown parameters, and E is an $N \times n$ matrix of error terms assumed to be multivariate normally distributed with mean 0 and variance covariance $\sigma^2 I$. Researchers in a study [9] estimated the unknown matrices of parameter as:

$$\hat{M} = (G^{/}KG)^{-}G^{/}KZLH(H^{/}LH)^{-} \quad (24)$$

$$\hat{B} = K^{-}KQ_{G/K}ZLH(H^{/}LH)^{-} \quad (25)$$

$$\hat{C} = (G^{/}KG)^{-}G^{/}KZQ^{/}_{H/L}LL^{-} \quad (26)$$

$$\hat{E} = P_{G/K}ZP^{/}_{H/L} - K^{-}KQ_{G/K}ZP^{/}_{H/L} - P_{G/K}ZQ^{/}_{H/L}LL^{-}$$

$$Q_{G/K} = I - P_{G/K}, P_{G/K} = G(G^{/}KG)^{-}G^{/}K$$

$$Q_{H/L} = I - P_{H/L}, P_{H/L} = H(H^{/}LH)^{-}H^{/}L$$

K, a symmetric nonnegative definite (nnd) matrix of order N denote the rows Metric matrix, and L, a symmetric nnd matrix of order n, to denote the columns metric matrix. If K and/or L are positive-semidefinite (psd) but not pd, the conditions: rank (K G) = rank (G), and rank (L H) = rank (H) has been required. These conditions were essential for the projection matrices [9]). When $K = I$ and $L = I$. Putting the estimates of $M, B, C, and\ E$ above in model (27) yields the following decomposition of the data matrix:

where:

$$Z = P_{G/K}ZP^{/}{}_{H/L} + K^{-}KQ_{G/K}ZP^{/}{}_{H/L} + P_{G/K}ZQ^{/}{}_{H/L}LL^{-} + (Z - P_{G/K}ZP^{/}{}_{H/L} - K^{-}KQ_{G/K}ZP^{/}{}_{H/L} - P_{G/K}ZQ^{/}{}_{H/L}LL^{-}) \quad (27)$$

Using the general model of restricted principal component:

$$\hat{Z}_{N.n} = G(G^{/}G)^{-1}G^{/}Z + \mathrm{GS}(G^{/}G)^{-1}H(H^{/}(G^{/}G)^{-1}H)^{-1}H^{/} - G(G^{/}G)^{-1}G^{/}Z\,H(H^{/}(G^{/}G)^{-1}H)^{-1}H^{/}\ (G^{/}G)^{-1}$$

let: $Z = GS$, $G = XT_r$, $L = L^{/} = (G^{/}G)^{-1}$, $H = T^{/}R^{/}$, and $RT_rS^{/} = k$.

Where S is a p x n matrix and: $S = (M_{p.q}H^{/}_{q.n} + C_{p.n})$.

$$\hat{Z}_{N.n} = G(G^{/}G)^{-1}G^{/}Z + \mathrm{G}(G^{/}G)^{-1}{T_r}^{/}R^{/}(RT_r(G^{/}G)^{-1}{T_r}^{/}R^{/})^{-1}RT_rS^{/} - G(G^{/}G)^{-1}{T_r}^{/}R^{/}(RT_r(G^{/}G)^{-1}{T_r}^{/}R^{/})^{-1}RT_r\ (G^{/}G)^{-1}G^{/}Z$$

$$= XT_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}X^{/}Y + XT_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/}(RT_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/})^{-1}RT_r\boldsymbol{S}^{/} - XT_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/}(RT_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/})^{-1}RT_r\ ({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}X^{/}Y$$

$$= X\hat{\beta}_{pc} + XT_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/}(RT_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/})^{-1}RT_r\boldsymbol{S}^{/} - XT_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/}(RT_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/})^{-1}RT_r{T_r}^{/}\ \hat{\beta}_{pc}$$

where:

$$T_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/}(RT_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/})^{-1}RT_r = I\ (USING\ RIGHT\ AND\ LEFT\ INVERSE)$$

$$\hat{Z}_{N.n} = X\hat{\beta}_{pc} + \mathrm{X}T_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/}(RT_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/})^{-1}(k - R\hat{\beta}_{pc}) = X\hat{\beta}^{C}_{pc}$$

$$\hat{\beta}^{C}_{pc} = \hat{\beta}_{pc} + T_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/}(RT_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/})^{-1}(k - R\hat{\beta}_{pc}) = \hat{\beta}_{PC} + T_r\Lambda_r^{-1}{T_r}^{/}R^{/}(RT_r\Lambda_r^{-1}{T_r}^{/}R^{/})^{-1}(k - R\hat{\beta}_{PC})$$

this is the same result as (17), then (18) can be used to get the proposed estimator:

$$\hat{\beta}_{MSRPC} = \hat{\beta}_{PC} + T_r{\Lambda_r}^{-1}T_r^{/}R^{/}(V + RT{\Lambda_r}^{-1}T_r^{/}R^{/})^{-1}(k - R\hat{\beta}_{PC})$$

3.4. The Fourth Method

In (1) putting $X = XT_r$, $R = RT_r$, and $\beta = {T_r}^{/}\beta$ and substitute in (6) $[\hat{\beta}_{OME} = \hat{\beta}_{OLS} + (X^{/}X)^{-1}R^{/}(V + R(X^{/}X)^{-1}R^{/})^{-1}(k - R\hat{\beta}_{OLS})]$:

$$({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}X^{/}Y + ({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/}(V + RT({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/})^{-1}(k - RT_r{T_r}^{/}\hat{\beta}_{OLS})$$

Multiply with T_r, the equation become:

$$T_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}X^{/}Y + T_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/}(V + RT_r({T_r}^{/}X^{/}XT_r)^{-1}{T_r}^{/}R^{/})^{-1}(k - RT_r{T_r}^{/}\hat{\beta}_{OLS})$$

$$= \hat{\beta}_{PC} + T_r\Lambda_r^{-1}{T_r}^{/}R^{/}(V + RT_r\Lambda_r^{-1}{T_r}^{/}R^{/})^{-1}(k - R\hat{\beta}_{PC}) = \hat{\beta}_{MSRPC} \quad (28)$$

4. Numerical Example

This example illustrate the performance of the proposed estimator, the real data example which was found in previous studies has been considered [3], where the sample size n=10, the number of independent variables p=5, k = Rβ + $\boldsymbol{\epsilon}^*$, k = -0.2685, R =(1, 1, 2, -2, -2), $\boldsymbol{\epsilon}^* \sim \mathcal{N}(0, \hat{\sigma}^2)$.

The R programme version 3.5.0 used to get the following results from it:

Table 1. *The parameter coefficients and the bias for each estimator.*

Parameter Estimator	b_0	b_1	b_2	b_3	b_4
$\hat{\beta}_{OLS}$	0.6921	0.6258	-0.1154	0.2866	0.0256
$\hat{\beta}_{OME}$	0.4014 (-0.5985)	0.5695 (-0.4305)	-0.1543 (-1.1543)	0.3402 (-0.6597)	0.1252 (-0.8748)
$\hat{\beta}_{PC}$	-0.0182 (-1.0181)	0.5359 (-0.4641)	-0.0344 (-1.0343)	0.2838 (-0.7162)	0.2104 (-0.7896)
$\hat{\beta}_{SRPC}$	0.0895 (-0.9104)	0.5568 (-0.4432)	-0.0199 (-1.0199)	0.2639 (-0.7360)	0.1735 (-0.8264)
$\hat{\beta}_{MSRPC}$	0.0327 (-0.9672)	0.2607 (-0.7393)	0.2023 (-0.7977)	0.3145 (-0.6855)	0.1688 (-0.8311)
$\hat{\beta}_{PC}$ (p = r)	0.6921 (-0.3079)	0.6258 (-0.3741)	-0.1154 (-1.1154)	0.2866 -0.7134)	0.0256 (-0.9743)
$\hat{\beta}_{MSRPC}$ (p = r)	0.4014 (-0.5985)	0.5695 (-0.4305)	-0.1543 (-1.1543)	0.3402 (-0.6597)	0.1252 (-0.8748)
$\hat{\beta}_{SRPC}$ (p = r)	0.4014 (-0.5985)	0.5695 (-0.4305)	-0.1543 (-1.1543)	0.3402 (-0.6597)	0.1252 (-0.8748)

The previous table shows the parameter coefficients and the bias for the least square estimator ($\hat{\beta}_{OLS}$), the ordinary mixed estimator ($\hat{\beta}_{OME}$), the principal component estimator ($\hat{\beta}_{PC}$), the stochastic restricted principal component ($\hat{\beta}_{SRPC}$) which introduced in previous studies [3]. The modified stochastic restricted principal component which introduced in this paper ($\hat{\beta}_{MSRPC}$), and the principal component estimator in case of p=r ($\hat{\beta}_{PC}$ (p = r)), the ($\hat{\beta}_{SRPC}$) in case of p = r, and the ($\hat{\beta}_{MSRPC}$) in case of p = r. The numbers between brackets represent the bias. The total mean square error (TMSE) criteria used to compare between the SRPC and the MSRPC. It represent the summation of the MSE for each estimator. It was 251.8384 for the SRPC and 8.6271 for the MSRPC. This is means that the new estimator (MSRPC) is better than the old estimator (SRPC).

The previous results show that both estimators the old ($\hat{\beta}_{SRPC}$) and the new ($\hat{\beta}_{MSRPC}$) are equivalent with the OLS estimator in case of r = p, the parameter coefficient, and the bias were the same in each case. A different results in case of r < p was found, where the TMSE in the old estimator was 251.8384 greater than that TMSE 8.6271 in the new estimator. This mean, that the MSRPC estimator is better than the SRPC estimator.

5. Application Case

This application illustrate the performance of the proposed estimator. The data represent 1058 units of air conditioner that sailed from July 2007 to March 2013 in an Egyptian company called Pure Technology. The data has been decomposed into 23 different cases. Where the number of independent variables p=6, the population parameter $\beta = (0.4, 0.45, 0.5, 0.3, 0.35, 0.6)$, k = Rβ + ϵ^*, k = 1.5, R =(1, 1, 1, 1, 1, 1), $\epsilon^* \sim \mathcal{N}(0, \hat{\sigma}^2)$. The data were as follows:

Table 2. *The count of sales units of air conditioner at different cases.*

No.	Profit/1000 (Y)	1.5 HP/b (x_1)	2.25Hp/b (x_2)	3Hp/b (x_3)	1.5 Hp/c (x_4)	2.25HP/c (x_5)	3HP/c (x_6)
1	527.69	17	6	13	52	32	26
2	244.4	3	0	0	3	1	2
3	787.4	0	0	1	12	6	3
4	517.43	30	15	7	47	21	21
5	894.5	6	1	5	6	6	1
6	1.56	0	0	0	0	3	1
7	345.4	1	0	1	4	0	3
8	261.831	0	0	0	0	2	6
9	-0.168	0	0	0	2	0	1
10	585.668	4	0	0	1	4	6
11	9.64	5	0	1	2	4	16
12	46.997	20	15	11	29	26	29
13	3.657	1	2	2	3	0	1
14	22.733	14	9	5	17	9	10
15	50.449	45	13	11	37	29	21
16	4.465	2	0	1	2	3	3
17	3.22	0	0	1	4	3	1
18	4.882	1	1	1	5	1	3
19	3.000	0	1	0	2	3	3
20	16.682	2	1	2	1	8	28
21	13.650	3	1	8	2	5	16
22	11.7042	21	2	2	7	11	8
23	22.232	9	5	4	12	28	62

(Collected from an Egyptian air conditioner Company called Pure Technology [10]).

where:

1. 1.5 HP/b represent the air condition with power 1.5 horse and it is hot and cold.
2. 2.25 HP/b represent the air condition with power 2.25 horse and it is hot and cold.
3. 3HP/b represent the air condition with power 3 horse and it is hot and cold.
4. 1.5 HP/c represent the air condition with power 1.5 horse and it is cold.
5. 2.25 HP/c represent the air condition with power 2.25 horse and it is cold.
6. 3 HP/c represent the air condition with power 3 horse and it is cold.

Table 3. *The correlation matrix between the variables.*

Variable	Y	x_1	x_2	x_3	x_4	x_5	x_6
Y	1.0000	0.8530	0.9020	0.9050	0.9462	0.9180	0.6090
x_1		1.0000	0.8529	0.7338	0.7959	0.7839	0.4121
x_2			1.0000	0.7731	0.8314	0.7868	0.5014
x_3				1.0000	0.8360	0.8391	0.5236
x_4					1.0000	0.8524	0.4446
x_5						1.0000	0.7809
x_6							1.0000

The previous table indicate that all independent variables have high correlations with the dependent variable. There are found a high correlations between the independent variables, this means that the multicollinearity problem has been found. To solve this problem (PCA) technique will be used. An addition information will be used to tell us that the summation of the profit for each type of the air conditioner is 1500 pound.

Table 4. *The parameter coefficients and the bias for each estimator.*

Parameter Estimator	b_1	b_2	b_3	b_4	b_5	b_6
$\hat{\beta}_{OLS}$	0.2324	0.5373	1.1376	0.5826	-0.2216	0.2760
$\hat{\beta}_{OME}$	0.2326 (-0.1674)	0.5365 (0.0865)	1.1367 (0.6367)	0.5829 (0.2829)	-0.2217 (-0.5717)	0.2761 (-0.3239)
$\hat{\beta}_{PC}$	0.2849 (-0.1151)	0.1389 (-0.3111)	0.1367 (-0.3633)	0.6294 (0.3294)	0.3117 (-0.0383)	0.1627 (-0.4373)
$\hat{\beta}_{SRPC}$	0.2849 (-0.1151)	0.1388 (-0.3112)	0.1365 (-0.3635)	0.6294 (0.3294)	0.3117 (-0.0383)	0.1628 (-0.4372)
$\hat{\beta}_{MSRPC}$	0.2849 (-0.1151)	0.1389 (-0.3111)	0.1367 (-0.3633)	0.6294 (0.3294)	0.3117 (-0.0383)	0.1627 (-0.4373)
$\hat{\beta}_{PC}$ (p = r)	0.2324 (-0.1676)	0.5373 (0.0873)	1.1376 (0.6376)	0.5826 (0.2826)	-0.2216 (-0.5716)	0.2760 (-0.3240)
$\hat{\beta}_{MSRPC}$ (p = r)	0.2326 (-0.1674)	0.5365 (0.0865)	1.1367 (0.6367)	0.5829 (0.2829)	-0.2217 (-0.5717)	0.2761 (-0.3239)
$\hat{\beta}_{SRPC}$ (p = r)	0.2326 (-0.1674)	0.5365 (0.0865)	1.1367 (0.6367)	0.5829 (0.2829)	-0.2217 (-0.5717)	0.2761 (-0.3239)

The previous table shows the parameter coefficients and the bias for the least square estimator ($\hat{\beta}_{OLS}$), the ordinary mixed estimator ($\hat{\beta}_{OME}$), the principal component estimator ($\hat{\beta}_{PC}$), the stochastic restricted principal component ($\hat{\beta}_{SRPC}$) [3]. The modified stochastic restricted principal component which introduced in this paper ($\hat{\beta}_{MSRPC}$). And the principal component estimator in case of p = r ($\hat{\beta}_{PC}$ (p = r)), the ($\hat{\beta}_{SRPC}$) in case of p = r, and the ($\hat{\beta}_{MSRPC}$) in case of p = r. The numbers between brackets represent the bias. The total mean square error (TMSE) criteria used to compare between the SRPC and the MSRPC. It represent the summation of the MSE for each estimator. It was 148.0862 for the SRPC and 147.9901 for the MSRPC. This is means that the new estimator (MSRPC) is better than the old estimator (SRPC). A simulation study have done to assess this result.

The previous results show that both estimators the old ($\hat{\beta}_{SRPC}$) and the new ($\hat{\beta}_{MSRPC}$) are equivalent with the OLS estimator in case of r = p, the parameter coefficient, and the bias were the same in each case. A different results in case of r < p was found, where the TMSE in the old estimator was 148.0862 greater than that TMSE 147.9901 in the new estimator. This mean, that the MSRPC estimator is better than the SRPC estimator. Moreover, this same result from the previous section has been got.

6. Simulation Study

A simulation study with 5000 replications has been done to check the results at different cases. The different cases of the sample size (n) was 10, 20, 30, 50, 100, and 200. The results of the restriction model (k) was 0, 1, and -1. The number of variables in the model (p) was 2, 3, 4, and 5. the number of components (r) were always less than the number of the variables (p). Constant values has been used for R, and the true parameters (β) where it consists of vector of $1_{1.p}$. Data was distributed multivariate normally with vector of means $\mu = 0_{1.p}$, and variance covariance matrix Σ. Σ has been chosen according to high correlations between variables (multicollinearity problem), it was:

$$\Sigma = \begin{pmatrix} 81 & 28.8 \\ & 16 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 100 & 63 & 76.5 \\ & 49 & 47.25 \\ & & 81 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 6561 & 65.232 & -55.755 & -10.544 \\ & 64 & 52.864 & 47.712 \\ & & 49 & 39.816 \\ & & & 144 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 81 & 64.152 & 56.826 & 40.635 & 82.26 \\ & 64 & 51.016 & 36 & 72 \\ & & 49 & 31.57 & 63 \\ & & & 25 & 45.1 \\ & & & & 100 \end{pmatrix}$$

TMSE was chosen as a criterion for all cases (180 case). The main steps for each case were:

1. Generate p independent variables which distributed multivariate normally with mean $\boldsymbol{\mu}$ and variance covariance matrix Σ. Generate one dependent variable, which distributed normally with mean 0 and variance 1.
2. Calculate the SRPC and the MSRPC estimators.
3. Calculate the TMSE criterion for each estimator.
4. Replicate the previous steps 5000 times.
5. Calculate the Mean of TMSE for each estimator.
6. Calculate the difference of the TMSE between the two estimators.

The results of the simulation has been indicated as follows in the next table:

Table 5. *The difference of the TMSE between the SRPC estimator and the MSRPC estimator.*

Estimator	**p, r n, k**		**p=2**	**p=3**		**p-4**		
			r=1	**r=1**	**r=2**	**r=1**	**r=2**	**r=3**
SRPC - MSRPC	n=10	k=0	4.28E-02	8.74E-01	2.63E-02	2.28E+05	-1.11E+03	1.44E-01
		k=1	1.54E-02	2.03E-02	2.26E-02	-3.68E+06	-1.54E+02	1.77E+00
		k=-1	7.24E-02	9.95E-01	4.73E-02	6.98E+07	-1.82E+05	1.02E+01
	n=20	k=0	1.49E-02	6.82E-03	6.16E-03	1.14E+05	3.35E+01	5.61E-03
		k=1	5.64E-03	4.44E-03	4.09E-03	-1.13E+06	-1.46E+01	4.15E-03
		k=-1	2.40E-02	9.33E-03	8.55E-03	3.78E+05	1.55E+02	7.22E-03
	n=30	k=0	8.79E-03	3.73E-03	3.51E-03	1.07E+06	3.71E+01	2.77E-03
		k=1	3.40E-03	2.41E-03	2.30E-03	-1.55E+06	-1.46E+00	2.05E-03
		k=-1	1.43E-02	5.06E-03	4.72E-03	-1.03E+06	-1.11E+01	3.46E-03
	n=50	k=0	4.91E-03	1.96E-03	1.88E-03	2.16E+04	2.48E-03	1.33E-03
		k=1	1.92E-03	1.28E-03	1.22E-03	-8.01E+05	1.81E-03	1.00E-03
		k=-1	7.85E-03	2.65E-03	2.55E-03	-1.19E+07	2.88E-03	1.70E-03
	n=100	k=0	2.34E-03	8.98E-04	8.64E-04	-1.94E+06	9.23E-04	5.83E-04
		k=1	9.21E-04	5.84E-04	5.66E-04	1.30E+06	6.64E-04	4.38E-04
		k=-1	3.73E-03	1.19E-03	1.17E-03	9.50E+07	1.17E-03	7.38E-04
	n=200	k=0	1.13E-03	4.26E-04	4.14E-04	-3.42E+05	4.24E-04	2.72E-04
		k=1	4.51E-04	2.78E-04	2.72E-04	-1.20E+04	3.09E-04	2.03E-04
		k=-1	1.81E-03	5.69E-04	5.56E-04	1.61E+05	5.41E-04	3.43E-04

Table 6. *Contiuned.*

Estimator	**p, r n, k**		**p=5**			
			r=1	**r=2**	**r=3**	**r=4**
SRPC - MSRPC	n=10	k=0	1.13E+03	1.46E+02	4.22E+02	1.89E+02
		k=1	5.16E+01	3.61E+01	5.98E+01	3.77E+01
		k=-1	6.57E+03	5.18E+02	9.91E+01	1.33E+02
	n=20	k=0	3.72E-02	3.26E-02	3.23E-02	2.47E-02
		k=1	2.79E-02	2.48E-02	2.28E-02	1.84E-02
		k=-1	5.04E-02	4.56E-02	3.82E-02	3.11E-02
	n=30	k=0	1.42E-02	1.20E-02	1.02E-02	8.26E-03
		k=1	1.09E-02	9.51E-03	7.90E-03	6.64E-03
		k=-1	1.74E-02	1.47E-02	1.23E-02	1.02E-02
	n=50	k=0	5.99E-03	4.71E-03	3.67E-03	3.06E-03
		k=1	4.69E-03	3.75E-03	2.89E-03	2.43E-03
		k=-1	7.25E-03	5.77E-03	4.48E-03	3.73E-03
	n=100	k=0	2.39E-03	1.70E-03	1.24E-03	9.57E-04
		k=1	1.90E-03	1.35E-03	9.69E-04	7.60E-04
		k=-1	2.92E-03	2.07E-03	1.48E-03	1.15E-03
	n=200	k=0	1.08E-03	7.03E-04	4.91E-04	3.53E-04
		k=1	8.59E-04	5.58E-04	3.92E-04	2.82E-04
		k=-1	1.31E-03	8.56E-04	5.91E-04	4.29E-04

The previous table indicate that in most of the cases, the SRPC estimator have values greater than the MSRPC estimator. Most of cases which have the opposite results found in (p = 4, k = 1). The number of these cases reach to 15 case. These opposite results may be found because of the negative correlations between the independent variables. So the MSRPC estimator is better than the SRPC estimator with percentage 91.67% $\left(\frac{180-15}{180}\right)$.

7. Summary

The stochastic restricted principal components (SRPC) regression estimator ignoring the number of components (orthogonal matrix T_r) that has been chosen to solve the multicollinearity problem in the data matrix (X). This paper introduced another estimator which uses matrix T_r to get more accurate results. The new estimator uses any number of components that have been required. A numerical example and an application were given to illustrate the performance of the proposed estimator. The previous results show that, the TMSE for the old estimator is greater than the TMSE on the new estimators, this is means that there was found more accuracy in making decision when using three components. Both estimators, the old and the new were equivalent with the OLS estimator in case of r = p, regarding the parameters coefficients, bias values, and the MSE. The simulation results indicate the same results that have been got from the numerical example (real data) and the application in many different cases.

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